The Theories and Principles behind Hardware Implementation of Any Point DFT

There are several commonly used forms of expression for DFT, only the constant factors in front of the DFT expressions are different in these various forms, other aspects are identical. In communications, the following is the usually used form for *N*-point DFT:

$$X(k)=\sqrt{\frac{1}{N}}\sum_{n=0}^{N-1}x(n)e^{-j2\pi k\frac{n}{N}} \tag{1}$$

Accordingly, *N*-point IDFT is expressed as:

$$x(n)=\sqrt{\frac{1}{N}}\sum_{k=0}^{N-1}X(k)e^{j2\pi k\frac{n}{N}} \tag{2}$$

From formulas (1) and (2), we know that the IDFT result can be obtained by DFT operation through feeding the complex conjugate of the input data to DFT operation and taking complex conjugate of the output data from DFT operation. Therefore, all of the DFT related discussions can be extended to IDFT.

In the following, it will be explained that if *M*>*N*, *N*-point DFT can be obtained precisely by *M*-point DFT and related signal processing methods. As we know, if $M=2^n$ or $M=3\times 2^n$ is selected, there are some FFT algorithms to implement *M*-point DFT. In other words, any point DFT can be realized in hardware by utilizing certain FFT processor. For convenience, the constant factor $1/\sqrt{N}$ in DFT/IDFT formula will be neglected in the subsequent analysis, and the conclusions will not be affected by this.

According to the theory of signal processing, discrete Fourier transform (DFT) is actually a portion of discrete Fourier series (DFS) of discrete periodic time signal in DFT window. Due to the issue of sampling rate conversion, using DFS is not convenient, in this paper we apply discrete time Fourier transform (DTFT) to demonstrate related theories and principles. Be aware that the spectrum line of DTFT is δ -function and its strength matches corresponding coefficient of DFS. In the following descriptions, for constant factors in various formulas, which have no influences on the properties of arithmetic operations, will not be considered.

Figure 1 shows a periodic continuous signal with period T_{sym} and a periodic discrete signal which is obtained from the continuous signal through sampling. For the discrete signal, there are *N* samples distributed evenly in each T_{sym} interval, and the sampling period is T_{s1}. the following relationship is established in figure 1:

$$N \cdot T_{s1} = T_{s\ y} \qquad (3)$$

$$x(n) = x(nT_{s1}), \qquad n = 0, 1, \cdots, N-1 \qquad (4)$$

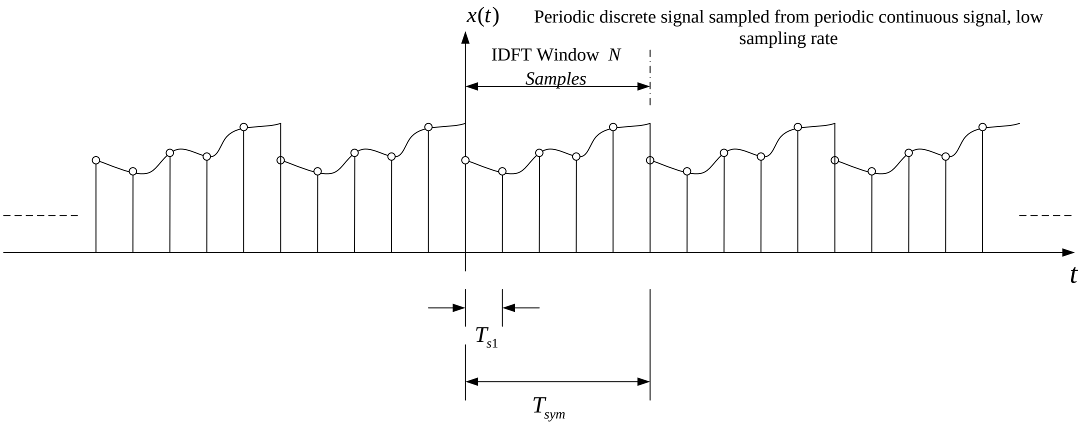

Figure 1

The DFT results of the N samples shown in the IDFT window of figure 1 are corresponding to the N points frequency response of DTFT shown in the DFT window of figure 2. As we know, the DFT results can be calculated through the following formula. Please note that the constant factor before Σ is neglected in this expression.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{kn}{N}}, \quad k = 0, 1, \cdots, N-1 \qquad (5)$$

the frequency difference between adjacent frequency indexes:

$$\Delta f_1 = \frac{1}{NT_{s1}} = \frac{1}{T_{sym}} \qquad (6)$$

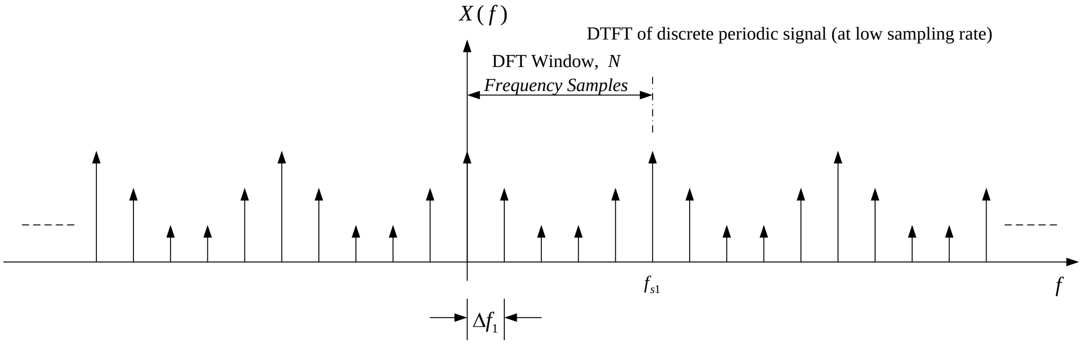

Figure 2

According to the sampling theorem, after the signal with its DTFT shown in figure 2 passing through a ideal low-pass filter shown in figure 3, the continuous periodic time signal shown in figure 1 will be restored. The coefficients of the remained impulses of the spectrum shown in figure 2 after passing through the low-pass filter shown in figure 3 can be obtained through the DFT expression stated in formula (5). Therefore, according to the features of Fourier transform, the original continuous-time periodic signal can be got without distortion by the following equation.

$$x(t) = \sum_{k=0}^{\left\lfloor \frac{N}{2} \right\rfloor} X(k) e^{j2\pi \frac{kt}{T_{sym}}} + \sum_{k=\left\lfloor \frac{N}{2} \right\rfloor+1}^{N-1} X(k) e^{j2\pi \frac{(k-N)t}{T_{sym}}} \qquad (7)$$

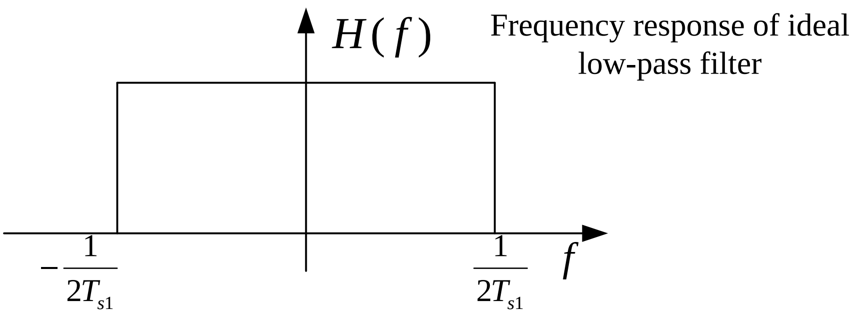

Figure 3

Figure 4 shows the reconstructed continuous-time periodic signal according to equation (7), which is identical to the continuous signal shown in figure 1. Now, the signal reconstructed from formula (7) is over sampled with respect to the samples in figure 1, and there are M ($M > N$) samples distributed evenly in each T_{sym} interval, the result after oversampling is also illustrated in figure 4.

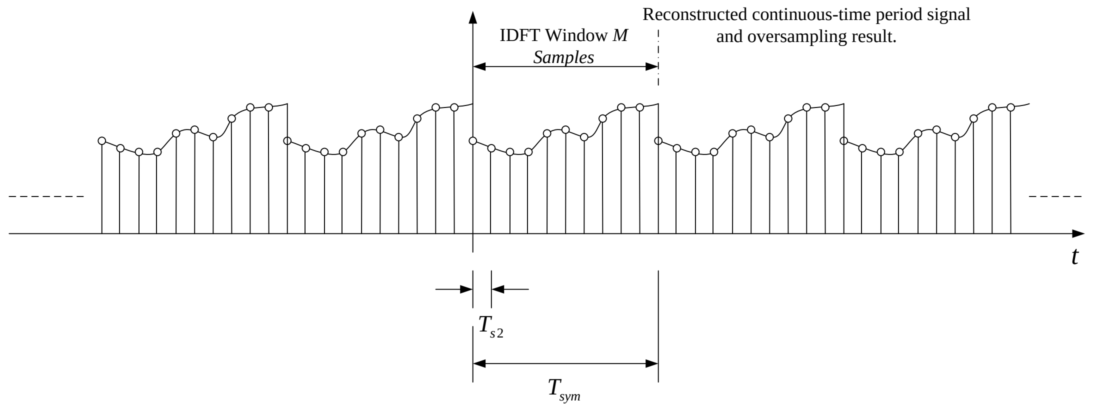

Figure 4

The time domain samples in IDFT window shown in figure 4 can be obtained through the following expressions:

$$M \cdot T_{s2} = T_{sym} \quad (8)$$

$$x'(m) = x(mT_{s2}) = \sum_{k=0}^{\left\lfloor \frac{N}{2} \right\rfloor} X(k) e^{j2\pi \frac{km}{M}} + \sum_{k=\left\lfloor \frac{N}{2} \right\rfloor+1}^{N-1} X(k) e^{j2\pi \frac{(k-N)m}{M}},$$

$$m = 0, 1, \cdots, M-1 \quad (9)$$

The DFT results of these M point $x'(m)$ are shown in the DFT window of figure 5, and the corresponding frequency difference between frequency indexes is expressed as follows.

$$\Delta f_2 = \frac{1}{MT_{s2}} = \frac{1}{T_{sym}} = \Delta f_1 \quad (10)$$

Therefore, there is a one-to-one relationship between the N spectrum lines in figure 2 and those in figure 5, and the frequency values of the corresponding frequency indices in the left side $\left\lfloor \frac{N}{2} \right\rfloor$ points are the same. $x'(m)$ can also be obtained by IDFT according to the following formula.

$$x'(m) = \sum_{k=0}^{M-1} X'(k) e^{j2\pi\frac{km}{M}}, \quad m = 0, 1, \cdots, M-1 \qquad (11)$$

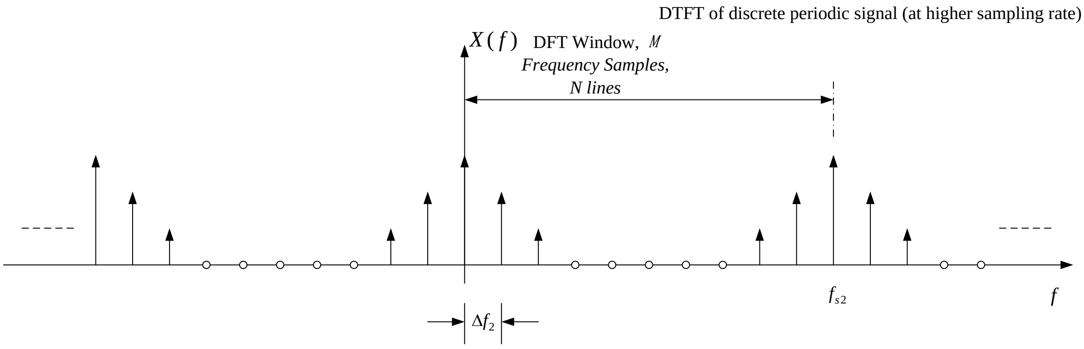

Figure 5

Data sequence $X'(k)$ is the *M*-point DFT result of $x'(m)$, formulas (11) and (9) express the same data sequence, let corresponding terms in these two formulas be equal, we can get the following relationship (note: the constant factors in front of DFT/IDFT formulas are neglected in the deduction process, this does not affect the correctness of results).

$$X(k) = \begin{cases} X'(k), & k = 0, 1, \cdots, \left\lfloor \frac{N}{2} \right\rfloor \\ X'(k-N+M), & k = \left\lfloor \frac{N}{2} \right\rfloor + 1, \cdots, N-1 \end{cases} \qquad (12)$$

where, $X(k)$ is the result of *N*-point DFT.

To sum up, *M*-point DFT can be used to calculate any *N*-point DFT in theory where *N* is less than *M*. If let $M = 2^n$ or $M = 3 \times 2^n$, FFT algorithms can be applied to realize *M*-point DFT. That is to say, through appropriate design, any point DFT can be realized by FFT algorithms.

According to the above theoretical analysis, Using *M*-point DFT to calculate *N*-point DFT requires infinite periodic expansion of *N* point input data and an ideal low-pass filter. In the actual process, the ideal low-pass filter is not possible, as well as infinite periodic expansion of the input data. But by some digital signal processing ways, we can let the result of calculating *N*-point DFT through *M*-point DFT approach the DFT result via formulas (1) or (5) infinitely.